



12.3 Diagonalization

Douglas Wilhelm Harder, LEL, M.Math.

dwharder@uwaterloo.ca

dwharder@gmail.com





Introduction

- In this topic, we will
 - See that we can write a matrix as a product of an orthogonal matrix, a diagonal matrix and the transpose of the orthogonal matrix
 - This is described as diagonalization
 - Look at a few examples
 - Consider some applications





Diagonalization

- Given a symmetric matrix $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$, then there exists an orthonormal basis of eigenvectors $\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_n$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$
 - Thus, $A\hat{\mathbf{p}}_k = \lambda_k \hat{\mathbf{p}}_k$ for all $k = 1, \dots, n$
- Define the matrix $P = (\hat{\mathbf{p}}_1 \cdots \hat{\mathbf{p}}_n)$
 - This is an orthogonal matrix,
for the columns form an orthonormal basis for $\mathbf{R}^n$
 - Recall that isometric matrices on real finite-dimensional vector spaces are called *orthogonal matrices*
 - Thus, $P^{-1} = P^T$,
and this is the only safe time it is to calculate the inverse





Diagonalization

- Now, given a vector $\mathbf{u}$, if we want to find scalars so that

$$\alpha_1 \hat{\mathbf{p}}_1 + \cdots + \alpha_n \hat{\mathbf{p}}_n = \mathbf{u}$$

then normally, we must solve

$$P\boldsymbol{\alpha} = \mathbf{u}$$

- However, because P is orthogonal, we simply have to find

$$\boldsymbol{\alpha} = P^{-1}\mathbf{u} = P^T\mathbf{u}$$

- Now, if $\alpha_1 \hat{\mathbf{p}}_1 + \cdots + \alpha_n \hat{\mathbf{p}}_n = \mathbf{u}$, then

$$\begin{aligned} A\mathbf{u} &= A(\alpha_1 \hat{\mathbf{p}}_1 + \cdots + \alpha_n \hat{\mathbf{p}}_n) \\ &= \alpha_1 A\hat{\mathbf{p}}_1 + \cdots + \alpha_n A\hat{\mathbf{p}}_n \\ &= \alpha_1 \lambda_1 \hat{\mathbf{p}}_1 + \cdots + \alpha_n \lambda_n \hat{\mathbf{p}}_n \end{aligned}$$





Diagonalization

- You will remember the definition of matrix-vector multiplication:

$$A\mathbf{u} = \alpha_1 \lambda_1 \hat{\mathbf{p}}_1 + \cdots + \alpha_n \lambda_n \hat{\mathbf{p}}_n$$

- The right-hand side is just the matrix $P = (\hat{\mathbf{p}}_1 \cdots \hat{\mathbf{p}}_n)$ multiplied by the vector

$$A\mathbf{u} = P \begin{pmatrix} \lambda_1 \alpha_1 \\ \vdots \\ \lambda_n \alpha_n \end{pmatrix}$$

- Also, the vector above can be found by multiplying the diagonal matrix of the corresponding eigenvalues

$$A\mathbf{u} = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^T \mathbf{u}$$





Diagonalization

- The consequence of this equation

$$A\mathbf{u} = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^T \mathbf{u}$$

is that every symmetric matrix can be written as a product of:

- An orthogonal matrix of normalized eigenvectors
- A diagonal matrix of the corresponding eigenvalues
- The transpose of the orthogonal matrix

$$A = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^T$$





Symmetry

- Note that this product is symmetric:

$$\begin{aligned}\left(PDP^T\right)^T &= \left(P^T\right)^T DP^T \\ &= PDP^T\end{aligned}$$





Diagonalization

- In order to create these matrices:
 - Given a symmetric matrix
 - Find all the eigenvalues for that matrix
 - For each eigenvalue, find a linearly independent basis for that eigenspace
 - That basis will likely not be orthogonal, so apply the Gram-Schmidt algorithm to create an orthonormal basis
 - This will yield n orthonormal vectors $\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_n$
 - Define the matrix $P = (\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_n)$
 - Let λ_k be the eigenvalue corresponding to the eigenvector $\hat{\mathbf{p}}_k$ and define

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$





Non-symmetric diagonalization

- Note that if a matrix has n linearly independent eigenvectors but is not symmetric, then the matrix P will not be orthogonal
 - Thus, calculating the inverse will necessarily require you to
 - Apply a numerically unstable algorithm to calculate the inverse, so that $A\mathbf{u} = PDP^{-1}\mathbf{u}$
 - Use Gaussian elimination and backward substitution, or some other matrix decomposition to solve $P\boldsymbol{\alpha} = \mathbf{u}$
 - Having found $\boldsymbol{\alpha}$, calculate $P\boldsymbol{\alpha}$
 - Because the eigenvectors are not orthogonal, you cannot rely that each will individually express the effect in the direction of that eigenvector





Example

- Find the diagonal decomposition of:

$$A = \begin{pmatrix} 0.08 & -1.44 \\ -1.44 & 0.92 \end{pmatrix} \quad \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 0.08 & 1.44 \\ 1.44 & \lambda - 0.92 \end{pmatrix} \\ = (\lambda + 1)(\lambda - 2)$$

$$\left(\begin{array}{cc|c} -1.08 & 1.44 & 0 \\ 1.44 & -1.92 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} -1.08 & 1.44 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \left(\begin{array}{cc|c} 1.92 & 1.44 & 0 \\ 1.44 & 1.92 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1.92 & 1.44 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\alpha_2 \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} \quad \hat{\mathbf{p}}_1 = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}$$

$$\alpha_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad \hat{\mathbf{p}}_2 = \begin{pmatrix} -0.6 \\ 0.8 \end{pmatrix}$$

$$P = \begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix} \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$





Example

- Find the diagonal decomposition of:

$$A = \begin{pmatrix} 10 & -2 & 2 \\ -2 & 13 & -4 \\ 2 & -4 & 13 \end{pmatrix} \quad \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 10 & 2 & -2 \\ 2 & \lambda - 13 & 4 \\ -2 & 4 & \lambda - 13 \end{pmatrix} \\ = (\lambda - 9)^2 (\lambda - 18)$$

$$\left(\begin{array}{ccc|c} -1 & 2 & -2 & 0 \\ 2 & -4 & 4 & 0 \\ -2 & 4 & -4 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} -1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$
$$\begin{pmatrix} 2\alpha_2 - 2\alpha_3 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \alpha_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \quad \hat{\mathbf{p}}_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ 1 \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} \quad \hat{\mathbf{p}}_2 = \begin{pmatrix} \frac{-0.4}{\sqrt{1.8}} \\ 0.8 \\ \frac{1}{\sqrt{1.8}} \\ 1 \end{pmatrix}$$





Example

- Find the diagonal decomposition of:

$$A = \begin{pmatrix} 10 & -2 & 2 \\ -2 & 13 & -4 \\ 2 & -4 & 13 \end{pmatrix} \quad \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 10 & 2 & -2 \\ 2 & \lambda - 13 & 4 \\ -2 & 4 & \lambda - 13 \end{pmatrix} \\ = (\lambda - 9)^2 (\lambda - 18)$$

$$\left(\begin{array}{ccc|c} 8 & 2 & -2 & 0 \\ 2 & 5 & 4 & 0 \\ -2 & 4 & 5 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 8 & 2 & -2 & 0 \\ 0 & 4.5 & 4.5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{pmatrix} 0.5\alpha_3 \\ -\alpha_3 \\ \alpha_3 \end{pmatrix} = \alpha_3 \begin{pmatrix} 0.5 \\ -1 \\ 1 \end{pmatrix}$$

$$\hat{\mathbf{p}}_3 = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$





Example

- Thus, we have that for

$$A = \begin{pmatrix} 10 & -2 & 2 \\ -2 & 13 & -4 \\ 2 & -4 & 13 \end{pmatrix}$$

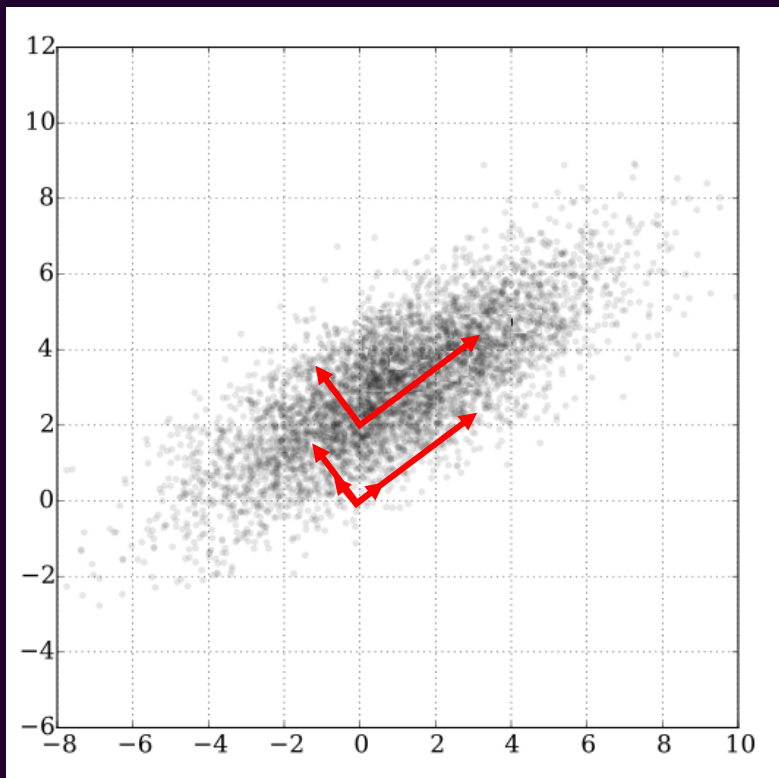
$$P = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-0.4}{\sqrt{1.8}} & \frac{1}{3} \\ \frac{1}{\sqrt{5}} & \frac{0.8}{\sqrt{1.8}} & -\frac{2}{3} \\ 0 & \frac{1}{\sqrt{1.8}} & \frac{2}{3} \end{pmatrix} \quad D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 18 \end{pmatrix}$$





Applications

- Consider this data:
 - There appears to be a relationship between the variables
 - Principal component analysis allows you to find this by:



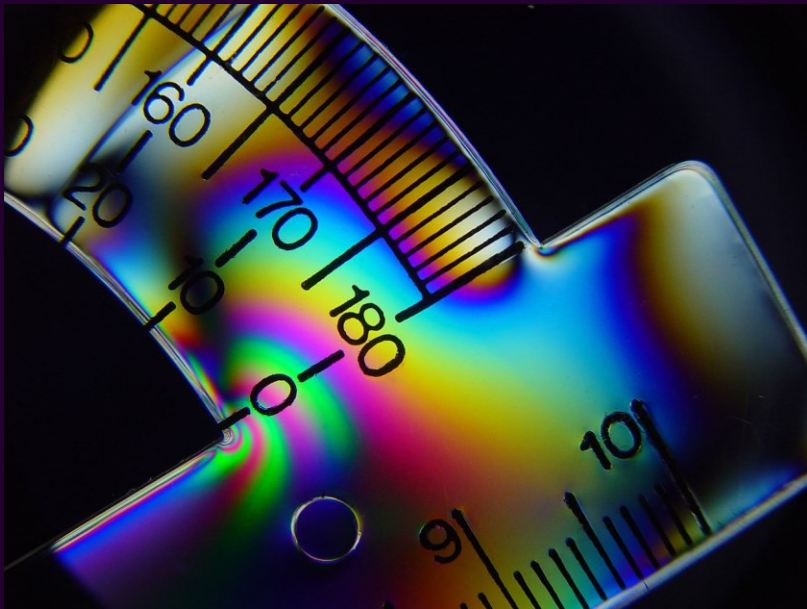
- Calculating the covariance matrix
- Finding the normalized eigenvectors
- Multiplying each eigenvector by its corresponding eigenvalue
- Transposing them to the center of mass





Applications

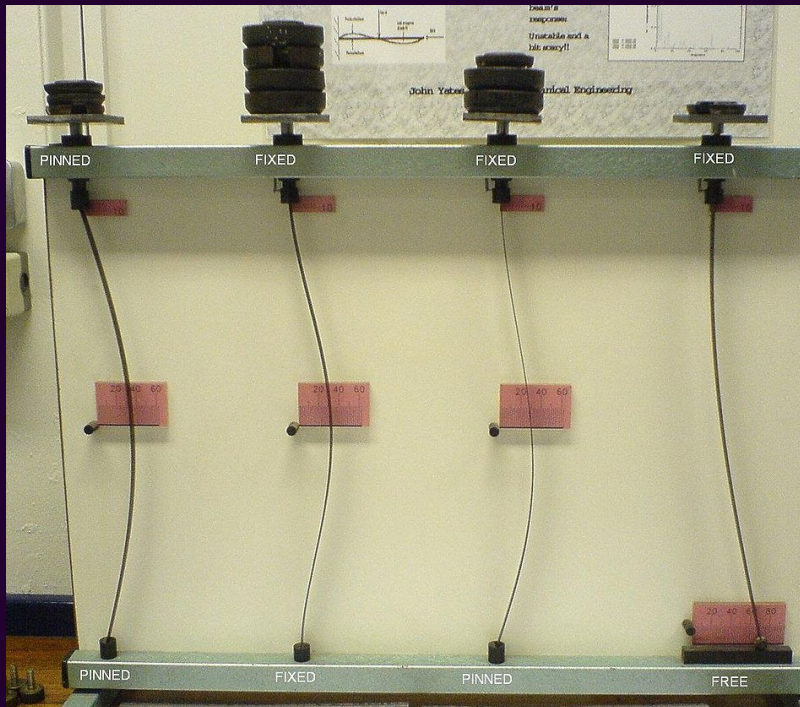
- Consider a mass that has multiple stresses placed up on it
 - The principal directions of the stress can be found by finding the eigenvalues and eigenvectors of the stress tensor
- Often the stress can be modeled on only the largest of these eigenvalues





Applications

- Given a load-bearing structure
 - When the structure is placed under a load, the structure not only bears the load, but also buckles under that load, thus changing its shape
- The relative effect of the buckling is representable by a symmetric tri-diagonal matrix:
 - Each point affects the points nearest it, and the effect on one point equals the effect on the other





Summary

- Following this topic, you now
 - Know that a symmetric matrix can be written as PDP^T
 - Understand this allows you to focus on effect of the matrix
 - Have seen two examples
 - Looked at a few applications





References

- [1] https://en.wikipedia.org/wiki/Diagonalizable_matrix#Diagonalization





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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